Effect of space charge on transverse instabilities in synchrotrons

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The effect of space charge on transverse instabilities in synchrotrons is considered. In addition to the coherent forces produced by image currents on the vacuum chamber walls, there are direct particle-particle forces that can significantly decrease the incoherent tune. Both types of space-charge force are a function of longitudinal position within a bunch, being proportional to the instantaneous current. In the low- and very-high-intensity regimes we find that the space-charge forces tend to reduce growth rates. A technique for the intermediate-intensity range, which includes the effect of incoherent space-charge tune spread, is introduced.

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I. INTRODUCTION

The problem of transverse instabilities in bunched beams has received much attention in the past. In the low-intensity limit, where the difference between the coherent and incoherent betatron tunes is small compared to the synchrotron tune, well established formalisms exist [1–5]. These formalisms yield expressions for the coherent betatron frequencies and the normal modes of the beam. When the space-charge tune shift is large compared to the synchrotron tune, the low intensity formulas usually yield coherent frequency shifts (CFS's) that are larger than the synchrotron frequency, which violates the assumptions in the derivation of the equations. The large CFS's are due to the transverse space-charge impedance, which is given by

$$Z_{\perp,\mathrm{SC}}(\omega)=i\frac{RZ_0}{\beta^2\gamma^2}\left[\frac{1}{a^2}-\frac{1}{b^2}\right], \tag{1}$$

where R is the machine radius, $\beta=v/c$, γ is the Lorentz factor, $Z_0=377\Omega$, a is the radius of a uniform equivalent beam, and b is the radius of the vacuum chamber. When $a \ll b$, the CFS's obtained from the low-intensity formulas are essentially weighted averages of the incoherent space-charge frequency shift. When the betatron tune shift is comparable to, or larger than, the synchrotron tune, the mode coupling formalism has been used [6–9,4,5]. Studies in the moderate coupling limit have usually been applied to high energy electron machines where the transverse impedance is often modeled as a resonator impedance,

$$Z(\omega)=\frac{R_\perp}{\dfrac{\omega}{\omega_r}+iQ_r\left[1-\dfrac{\omega^2}{\omega_r^2}\right]}, \tag{2}$$

where $R_\perp$ is the transverse shunt impedance of the resonator, Q_r is the quality factor, and ω_r is the angular resonant frequency. For a broad band resonator impedance, it is generally found that the growth rate of the most unstable mode increases dramatically when the current rises above the threshold where the coherent frequencies of two neighboring modes cross. This is referred to as the mode coupling instability.

This paper considers the transverse stability problem for various sizes of the space-charge tune shift. In Sec. II, the equations used in the moderate coupling limit are reviewed, and an upper limit to the growth rate for a Gaussian beam is derived. We consider the simplest, nontrivial case of mode coupling analytically and find that the space-charge impedance may affect the threshold of the instability, but that the magnitude of the growth rate is governed by the resistive impedance. In Sec. III, the effect of the space-charge tune spread is considered. First, the case of an unperturbed beam with constant radius, parabolic line density, and the associated incoherent space-charge frequency spread is considered. The Vlasov equation is reduced to a matrix equation for this case. The solution of the lowest order synthetic kernel approximation, in the weak coupling limit, shows that incoherent space charge may lead to Landau damping if the broad band transverse impedance is large enough. This section also includes a "computer friendly" technique for the case of a bunch with constant line density. Section III B considers the case where the unperturbed longitudinal distribution is Gaussian; the space-charge tune spread is retained. The perturbation is expanded in Hermite polynomials and the differential equation is reduced to a matrix equation. The system is solved by truncating the matrix and numerically solving the eigenvalue problem. Section III C involves the limit where the synchrotron tune is negligible compared to the betatron tune spread, and synchrotron motion is ignored. In this limit, the incoherent space-charge force can be handled exactly. The eigenvalue problem is solved for the case where the impedance consists of a single narrow band resonator and space charge. It is found that space charge can dramatically reduce growth rates. Our conclusions are summarized in Sec. IV.

II. MODERATE COUPLING

In this section, first order perturbation theory on the Vlasov equation is used to obtain a dispersion relation for

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the transverse dipole modes. We begin by considering the equations of motion when intensity dependent effects are neglected.

For the longitudinal motion, let θ denote S/R where R is the machine radius and S is the longitudinal Frenet-Serret coordinate. The angular revolution frequency is ω_0, and the longitudinal equation of motion is given by

$$\ddot{\phi}+\frac{\omega_s^2}{h}\sin(h\phi)=0 \;, \tag{3}$$

where a dot denotes a derivative with respect to time (t), and $\phi=\theta-\omega_0 t$ is angular phase measured with respect to the synchronous coordinates, ω_s is the synchrotron frequency, and h is the harmonic number. Let y denote the transverse coordinate of interest. We assume constant lattice functions and neglect horizontal to vertical coupling. The transverse equation of motion is [4]

$$\ddot{y}+\omega_y^2[1+(\xi-\eta)\delta]^2 y=\frac{\dot{y}\dot{\delta}(\xi-\eta)}{1+\delta(\xi-\eta)} \;, \tag{4}$$

where ω_y is the angular betatron frequency for an on-momentum particle, η is the frequency slip factor, ξ is the chromaticity, and δ is the fractional momentum deviation. For horizontal motion, one could include the momentum induced width by making the substitution $y\rightarrow y-\delta R/\gamma_t^2$ in Eq. (4). The fractional momentum deviation is related to the longitudinal coordinate via $\dot{\phi}=-\omega_0[\eta\delta+O(y^2/R^2)]$, and we neglect the correction that depends on y.

Let $\Psi(y,\dot{y},\phi,\dot{\phi},t)$ denote the phase space density. There are M bunches in the ring that are identical for no instability. Assume a solution of the form

$$\Psi_s=\sum_{k=0}^{M-1}\{\Psi_0(\phi-2\pi k/M,\dot{\phi},y,\dot{y}) +\exp(2\pi iks/M-i\Omega t) \times\Psi_1(\phi-2\pi k/M,\dot{\phi},y,\dot{y})\} \;, \tag{5}$$

where $s=0,1,\ldots,M-1$ is the coupled bunch mode number, and Ψ_1, which depends on s, is small compared to Ψ_0. Only the real part of the distribution has physical meaning. The transverse coherent force is driven by the dipole moment of the beam. Any resonating structure with a fixed position in the ring will be driven by the dipole moment as a function of azimuth and time $D(\theta,t)$. The normalization is defined by $D(\theta,t)=\langle y(\theta,t)\rangle\rho(\theta,t)$, where $\langle y(\theta,t)\rangle$ is the average offset of the beam and $\rho(\theta,t)$ is the line density of the particles. For the solution of the Vlasov equation, the natural longitudinal coordinate is ϕ, since the distribution oscillates only with frequency Ω in this coordinate. If we define the dipole eigenfunction in the beam frame by

$$D(\phi)=\int d\dot{\phi}\,dy\,d\dot{y}\;\Psi_1(\phi,\dot{\phi},y,\dot{y})y \;, \tag{6}$$

then the dipole moment in the ring frame is given by

$$\begin{aligned}D(\theta,t)&=\int_0^{2\pi}d\phi\,D(\phi)e^{-i\Omega t}\delta(\theta-\phi-\omega_0 t)\\&=\sum_k D_k\exp[i(kM+s)(\theta-\omega_0 t)-i\Omega t]\\&=D(\phi)e^{-i\Omega t} \;,\end{aligned} \tag{7}$$

where D_k is the Fourier component of $D(\phi)$ for harmonic $kM+s$. In addition to the coherent force, there is also an incoherent intensity dependent force that produces the incoherent space-charge tune depression. This will be neglected for now and considered in Sec. III. The coherent force is related to the dipole moment using a transverse wake potential in the smooth approximation,

$$\begin{aligned}F_\perp(\theta,y,t)&=\frac{q^2\omega_0}{2\pi R}\int_0^\infty d\tau\,W_\perp(\tau)D(\theta,t-\tau)\\&=\frac{iq^2\omega_0^2}{2\pi c}\sum_{k=-\infty}^{\infty}D_k Z_\perp[(kM+s)\omega_0+\Omega]\\&\quad\times\exp[i(kM+s)\phi-i\Omega t] \;,\end{aligned} \tag{8}$$

where q is the charge on a single particle, $W_\perp$ is the wake potential, and $Z_\perp$ is the transverse impedance. Notice that the force in the second expression is given in the beam frame coordinates and oscillates only at frequency Ω there. However, in the ring frame, all the betatron sidebands can drive the impedance. The expression for D_k may be obtained by integrating over a single bunch,

$$D_k=\frac{M}{2\pi}\int_{-\pi/M}^{\pi/M}D(\phi)\exp[-i(kM+s)\phi]d\phi \;. \tag{9}$$

The problem is reduced to studying the behavior of a single bunch, which we take to be centered at $\phi=0$.

Consider the amplitude angle variables defined implicitly by

$$\begin{aligned}y&=A\cos\chi \;,\\ \dot{y}&=-\omega_\perp(\delta)A\sin\chi \;,\\ \phi&=r\cos\psi \;,\\ \dot{\phi}&=-\omega_s r\sin\psi \;.\end{aligned}$$

The longitudinal variables follow from the equations of motion with the approximation $\sin(h\phi)\approx h\phi$. The transverse variables are approximate; they satisfy the constraint that the amplitude of the y oscillation does not depend on time, which follows from Eq. (4). The unperturbed distribution is given by $\Psi_0=L_0(r)T_0(A)$. The normalization in the new variables is taken as

$$\int_0^\infty L_0(r)r\,dr=\frac{1}{2\pi} \;,$$

$$\int_0^\infty T_0(A)A\,dA=\frac{N_t}{2\pi M} \;,$$

where N_t is the total number of particles in the ring, and we have absorbed factors of the oscillation frequencies. The Vlasov equation in first order perturbation theory is approximated by

$$-i\Omega\Psi_1+\omega_\perp(\delta)\frac{\partial\Psi_1}{\partial\chi}+\omega_s\frac{\partial\Psi_1}{\partial\psi}+\dot{A}\frac{\partial\Psi_0}{\partial A}=0\ . \tag{10}$$

The momentum dependence of the betatron frequency is taken to first order and is given by

$$\omega_\perp(\delta)=\omega_y+rQ_y\omega_s\left[\frac{\xi}{\eta}-1\right]\sin\psi\ , \tag{11}$$

where Q_y is the tune of an on-momentum particle. The coherent force is present in $\dot{A}$, which is given by

$$\dot{A}=-\frac{\sin\chi}{\gamma m\omega_y}F_\perp(r\cos\psi)\ , \tag{12}$$

where the time dependence has been divided out of $F_\perp(\phi)$, and the momentum dependence of the betatron frequency is not included in the coherent force. Next, we approximate the solution of Eq. (10) as

$$\Psi_1=g_1(r,\psi)\exp(i\chi)dT_0/dA\ ,$$

which amounts to neglecting the coupling between the betatron sidebands and expanding around the upper sideband $\Omega\approx\omega_y$. The accompanying approximation is to take $\sin\chi\approx -i\exp(i\chi)/2$, in $\dot{A}$. The equation for the distribution becomes

$$-i\Omega g_1+i\omega_\perp(\delta)g_1+\omega_s\frac{\partial g_1}{\partial\psi}=\frac{-i}{2\gamma m\omega_y}F_\perp(\phi)L_0(r)\ . \tag{13}$$

The coherent force is given by Eq. (8) with

$$D_k=-\frac{N_t}{2\pi}\int_0^\infty r\,dr\int_0^{2\pi}d\psi\,g_1(r,\psi)\exp[-i(kM+s)r\cos\psi]\ . \tag{14}$$

At this point, all the transverse variables have been removed, and we are let with an equation that involves the longitudinal variables alone. Next, the momentum dependence of the betatron frequency is removed by substituting

$$g_1=g_2\exp[iQ_y(\xi/\eta-1)r\cos\psi]\ ,$$

which results in

$$-i(\Omega-\omega_y)g_2+\omega_s\frac{\partial g_2}{\partial\psi}=\frac{-i\exp[-iQ_y(\xi/\eta-1)r\cos\psi]}{2\gamma m\omega_y}F_\perp(\phi)L_0(r)\ . \tag{15}$$

Dividing Eq. (15) by ω_s yields an equation of the form

$$iQg_2(r,\psi)+\frac{\partial g_2}{\partial\psi}=\tilde{F}(r,\psi)\ , \tag{16}$$

where $\tilde{F}$ is equal to the right hand side of Eq. (15) divided by ω_s, and $Q=(\omega_y-\Omega)/\omega_s$. The boundary condition is that g_2 is a periodic function of ψ. The solution of Eq. (16) is given by,

$$g_2(r,\psi)=\frac{1}{e^{2\pi iQ}-1}\int_0^{2\pi}d\psi' e^{iQ\psi'}\tilde{F}(r,\psi+\psi')\ , \tag{17}$$

which is easily verified by direct substitution. At this point, one can concentrate on solving for g_2 or on solving for the dipole harmonics. The final result will be identical in either case, and we will take the second option [4]. Given the expression (17) for g_2, and the relationship of g_1 to g_2, Eq. (14) is used to transform the integral equation into a matrix equation. The kernel is simplified using the Bessel generating function,

$$e^{ix\sin\theta}=\sum_{k=-\infty}^{\infty}J_k(x)e^{ik\theta}\ .$$

The final result is

$$D_m=\sum_{n=-\infty}^{\infty}T_{mn}D_n\ , \tag{18}$$

where the matrix is given by,

$$T_{mn}=\frac{icq\bar{I}Z_\perp[(nM+s)\omega_0+\Omega]}{2E_T\omega_sQ_y}\int_0^\infty r dr\,L_0(r)\sum_{\mu=-\infty}^{\infty}\frac{1}{Q+\mu}J_\mu(\tilde{n}r)J_\mu(\tilde{m}r)\ . \tag{19}$$

In Eq. (19), $E_T=\gamma mc^2$, $\bar{I}$ is the average (dc) current, $\tilde{n}=nM+s-Q_y(\xi/\eta-1)$, and $\tilde{m}$ is defined in the same way as $\tilde{n}$. Note that

$$Q=(\omega_y-\Omega)/\omega_s$$

contains the coherent frequency and plays the part of an eigenvalue.

At this point we will consider the special case of a Gaussian unperturbed distribution,

$$L_0(r)=\frac{e^{-r^2/2\sigma^2}}{2\pi\sigma^2} . \tag{20}$$

For this case, the integral in Eq. (19) can be found in standard tables [10]. The expression for the matrix element becomes

$$T_{mn}=i\chi_n(\Omega)e^{-(\tilde{m}^2+\tilde{n}^2)\sigma^2/2}\sum_{\mu=-\infty}^{\infty}\frac{I_\mu(\tilde{n}\tilde{m}\sigma^2)}{Q+\mu} , \tag{21}$$

where $I_\mu(x)$ is the modified Bessel function and

$$\chi_n(\Omega)=\frac{cq\bar{I}Z_\perp[(nM+s)\omega_0+\Omega]}{4\pi E_T\omega_s Q_y} . \tag{22}$$

We know of no exact solutions to Eq. (18) with matrix elements given by Eq. (21), and we will need to use various approximations in solving it. Before proceeding with the approximations, we establish an upper limit on the magnitude of the growth rate. Starting with Eq. (18), we find

$$|D_m|\le\sum_n|T_{mn}||D_n|\le\sup_k|D_k|\sum_n|T_{mn}| . \tag{23}$$

We assume that the spectrum of dipole harmonics is bounded, so that its supremum is finite. It follows that

$$\begin{aligned}\frac{|D_m|}{\sup_k|D_k|}&\le\sum_n|T_{mn}|\\&\le\sum_n|\chi_n|e^{-(\tilde{m}^2+\tilde{n}^2)\sigma^2/2}\sum_{\mu=-\infty}^{\infty}\frac{I_\mu(|\tilde{n}\tilde{m}\sigma^2|)}{|Q+\mu|}\\&\le\frac{\sup_n|\chi_n|}{\inf_\mu|Q+\mu|}\sum_n\exp[-(\tilde{m}^2+\tilde{n}^2)\sigma^2/2+|\sigma^2\tilde{m}\tilde{n}|] ,\end{aligned} \tag{24}$$

where we have used a summation theorem for the modified Bessel functions [10]. Since

$$e^{-(x^2+y^2)+2|xy|}\le e^{-(x+y)^2}+e^{-(x-y)^2} ,$$

for real x and y, the right hand side of the inequality can be made independent of m. Taking the supremum of the left hand side over m and simplifying gives

$$\inf_\mu|Q+\mu|\le\sup_n|\chi_n|2\sum_n e^{-n^2M^2\sigma^2/2}\approx\sup_n|\chi_n|\frac{\sqrt{8\pi}}{\sigma M} . \tag{25}$$

In practice, inequality (25) may be used to obtain a quick upper limit on growth rates in Gaussian beams. More accurate estimates require approximate solutions and are considered next.

A. Weak coupling limit

Before proceeding to the mode coupled case it will be instructive to examine the low-intensity limit. In this limit, $|\chi_n|\ll\sigma M$ for all n. Therefore, the solution requires that $|Q+\mu|\approx 0$ for some μ, and that $\Omega\approx\omega_y+\mu\omega_s$. This approximation is used for Ω in the evaluation of the impedance. Only the term in the sum with the small denominator is kept. Additionally, the modified Bessel function is approximated by its leading order term,

$$I_\mu(\tilde{n}\tilde{m}\sigma^2)\approx\frac{(\tilde{n}\tilde{m}\sigma^2)^{|\mu|}}{2^{|\mu|}|\mu|!} . \tag{26}$$

When Eq. (26) is substituted into Eq. (21), one finds that $D_m=(\tilde{m}\sigma)^{|\mu|}\exp(-\tilde{m}^2\sigma^2/2)$ is the eigenvector. Substituting this solution and solving for the coherent frequency gives

$$\Omega=\omega_y+\mu\omega_s-\frac{i\omega_s}{2^{|\mu|}|\mu|!}\sum_n\chi_n(\omega_y+\mu\omega_s)e^{-\tilde{n}^2\sigma^2}(\tilde{n}\sigma)^{2|\mu|} . \tag{27}$$

Usually, the offset by $\mu\omega_s$ in the evaluation of the impedance is negligible, and one uses $\chi_n(\omega_y)$. For the cases of interest, the frequency shift is large enough so that including the synchrotron offset is misleading. Expression (27) is the same as that obtained using the lowest order Besnier polynomial expansion [7] and is essentially the same as the expression given by Sacherer [1].

Higher order modes must be included when approximation (26) is not justified. This is the case when there is a significant impedance at frequencies higher than ω_0/σ and needs to be considered even if the frequency shift is small compared to the synchrotron tune. For this case, one still retains a single value of μ but includes higher order terms in the expansion for the Bessel function. Such a treatment is justified in studies of longitudinal stability when the frequency shift is small compared to the synchrotron tune, and space charge contributes to the high frequency impedance. However, it is not justified for the transverse case when $|\chi_n|\gtrsim 1$. The latter is of particular interest and will be studied in the following.

B. Solution with moderate coupling

When $|\chi_n| \gtrsim 1$, a few values of μ can significantly contribute to the sum. For this case, it is advantageous to rearrange the sum involving the Bessel functions,

$$\sum_\mu \frac{I_\mu(\bar{n}\bar{m}\sigma^2)}{Q+\mu} = \sum_{l=0}^{\infty} a_l(Q)(\bar{n}\bar{m}\sigma^2)^l . \tag{28}$$

In general,

$$a_l(Q) = \sum_{\mu=-l}^{l} \frac{1}{\mu+Q} \frac{1}{2^l[(l-\mu)/2]![(l+\mu)/2]!} , \tag{29}$$

where the sum is over even μ if l is even and over odd μ if l is odd. The first few coefficients are given by

$$a_0(Q) = \frac{1}{Q} , \quad a_1(Q) = \frac{Q}{Q^2-1} , \quad a_2(Q) = \frac{1}{2}\frac{Q^2-2}{Q(Q^2-4)} . \tag{30}$$

To proceed with the solution, the sum on the right-hand side of Eq. (18) is truncated at $l = l_{\max}$. Substituting this expression into Eq. (21) implies that D_m may be written as a linear combination

$$D_m = \sum_{l=0}^{l_{\max}} \alpha_l(\bar{m}\sigma)^l e^{-\bar{m}^2\sigma^2/2} , \tag{31}$$

where the α_l are unknown coefficients. Defining the force coefficients

$$f_l = i\sum_n \chi_n(\omega_y)(\bar{n}\sigma)^l e^{-\bar{n}^2\sigma^2} , \tag{32}$$

the equation for the coherent frequency and eigenvector is given by

$$\alpha_p = a_p(Q) \sum_{l=0}^{l=l_{\max}} \alpha_l f_{l+p} . \tag{33}$$

An analytic solution of Eq. (33) is possible for $l_{\max} \le 1$. For $l_{\max}=0$, one recovers the $\mu=0$ synchrotron mode in the low-intensity limit. For $l_{\max}=1$ there are two coupled equations,

$$\begin{bmatrix} (a_0 f_0 - 1) & a_0 f_1 \\ a_1 f_1 & (a_1 f_2 - 1) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = 0 .$$

The determinant of the matrix must vanish for a nontrivial solution. Substituting the expressions for a_0 and a_1 from Eq. (30), one obtains a polynomial constraint on Q,

$$(Q-f_0)(Q^2-1-Qf_2)-Qf_1^2=0 . \tag{34}$$

The solutions depend on the magnitude and phase of the force coefficients.

For electron machines, the mode coupling is found to be nearly independent of chromaticity [8], and the usual approach is to set $\xi=0$ in the mode coupling formula. Additionally, the source of impedance is thought to be broadband, and the characteristic frequency of the bunch spectrum is large compared to the bunching frequency. This allows the summation in Eq. (32) to be replaced by an integral. Under these circumstances, the force coefficients for even k are real and proportional to the imaginary part of the transverse inpedance. For odd k, f_k is imaginary and proportional to the real part of the impedance. Consider Eq. (34) under these circumstances. If $f_1=0$, the solutions are given by $Q=f_0, f_2/2 \pm \sqrt{1+f_2^2/2}$. The solutions for Q are real, even though they can cross as the force coefficients become large. Generally, when f_0, f_2, and f_1^2 are real, $|\mathrm{Im}(Q)| \le |f_1|$, which is proved in Appendix A. The maximum growth rate is proportional to the resistance, but the instability threshold depends on the reactance as well.

For many applications, the values of χ_n need to be computed numerically. Also, the dispersion relation for Q becomes very complicated as $l_{\max}$ increases. In this case, it is advantageous to define an additional index, μ corresponding to the synchrotron modes. The old index is related to the new indices via $l=|\mu|+2k$. After some rearrangement, Eq. (33) becomes

$$Q\alpha_{\mu,k} = -\mu\alpha_{\mu,k} + \sum_{|m|+2p\le l_{\max}} \alpha_{m,p} \sum_n \frac{i\chi_n e^{-\bar{n}^2\sigma^2}(\bar{n}\sigma)^{|m|+|\mu|+2(k+p)}}{k!(|\mu|+k)!2^{|\mu|+2k}} , \tag{35}$$

which is equivalent to Chin's result [9]. By defining an array $n(\mu,k)$, which is one to one with the ordered pairs (μ,k), Eq. (35) becomes a standard matrix equation and a computer code has been written to solve it. As an example, consider the case of space charge and a broadband resonator with $Q_r=1$, $\omega_r=\omega_0/\sigma$, and a shunt impedance such that $f_1=i0.2$. The space-charge tune shift is quantified using the value of f_0 obtained for space charge alone. Plots of the real parts of the resonant frequencies as a function of this variable are shown in Figs. 1–4. The onset of the instability occurs when two neighboring curves meet. The value of space-charge tune shift, at which the system becomes unstable, is relatively insensitive to $l_{\max}$. However, as tune shift increases, the solutions can switch between stable and unstable several

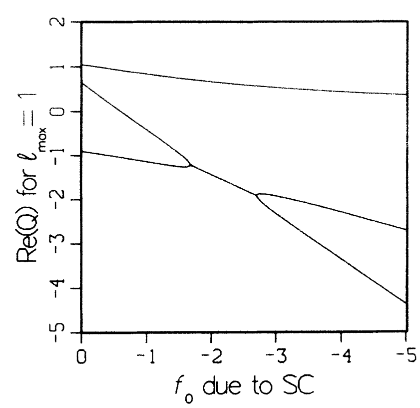

FIG. 1. Re(Q), for $l_{\max}=1$ as a function of incoherent space-charge tune shift.

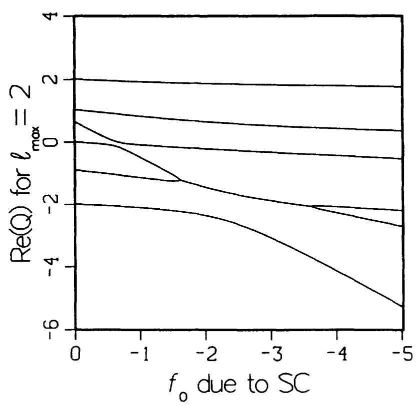

FIG. 2. Re(Q), for $l_{max}=2$ as a function of incoherent space-charge tune shift.

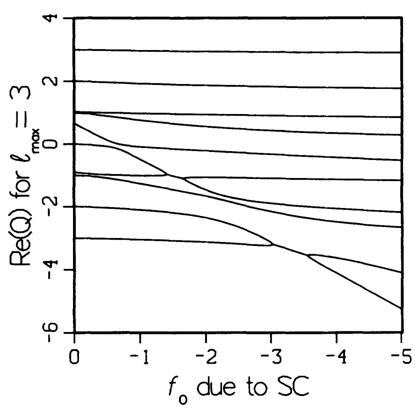

FIG. 3. Re(Q), for $l_{max}=3$ as a function of incoherent space-charge tune shift.

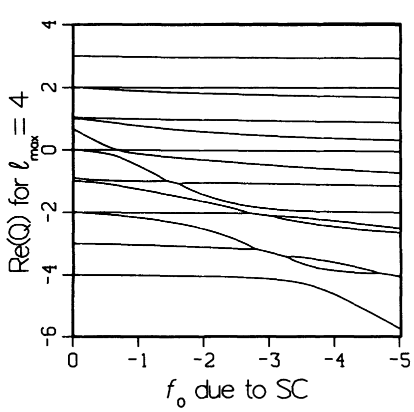

FIG. 4. Re(Q), for $l_{max}=4$ as a function of incoherent space-charge tune shift.

times. The detailed behavior of the solutions depends strongly on l_{max} but the peak growth rates, as a function of space-charge tune shift, are usually less sensitive to l_{max}.

When the chromaticity, discreteness, etc., are taken into account the solutions always have an imaginary part, but the growth rate can depend strongly on the space-charge tune shift. By examining the behavior of the growth rate as a function of tune shift, it appears that some useful upper limits can be found, but no threshold information. We go on to consider a more complete physical model that includes threshold information.

III. EFFECT OF SPACE-CHARGE TUNE SPREAD

In Sec. II, Eq. (11) was used for the incoherent betatron frequency. In the presence of a large space-charge tune spread this equation needs to be modified,

$$\omega_\perp(\delta,\phi)=\omega_y+rQ_y\omega_s\left[\frac{\xi}{\eta}-1\right]\sin\psi-\Delta\omega_{SC}\rho(\phi)/\rho(0)\ , \tag{36}$$

where $\Delta\omega_{SC}$ is the space-charge frequency depression in the center of the bunch, $\rho(\phi)$ is the line density, and it has been assumed that the radius of the beam is independent of ϕ. Under these conditions, Eq. (13) becomes

$$-i\Omega g_1+i\omega_\perp(\delta,\phi)g_1+\omega_s\frac{\partial g_1}{\partial\psi}=\frac{-i}{2\gamma m\omega_y}F_\perp(\phi)L_0(r)\ . \tag{37}$$

A. Parabolic line density

In general, it is difficult to obtain a matrix equation when the betatron frequency is given by Eq. (36). For a parabolic line density with $|\phi|\le\hat{\phi}$,

$$\rho(\phi)=\frac{3}{4\hat{\phi}}\left[1-\frac{\phi^2}{\hat{\phi}^2}\right]\ , \tag{38}$$

and some simplification is possible. The transverse betatron frequency is given by

$$\omega_\perp(\delta,\phi)=\omega_y+rQ_y\omega_s\left[\frac{\xi}{\eta}-1\right]\sin\psi-\Delta\omega_{SC}(1-\phi^2/\hat{\phi}^2)\ .$$

The ψ dependence of the betatron frequency is removed from Eq. (37) by the substitution

$$\begin{aligned}g_1&=g_2\exp\left[iQ_y\left[\frac{\xi}{\eta}-1\right]r\cos\psi-i\frac{\Delta\omega_{SC}}{4\hat{\phi}^2\omega_s}r^2\sin(2\psi)\right]\\&=g_2\exp[i\Phi(r,\psi)]\ .\end{aligned} \tag{39}$$

The equation for g_2 is given by

$$-i[\Omega-\omega_y+\Delta\omega_{SC}(1-r^2/2\hat{\phi}^2)]g_2+\omega_s\frac{\partial g_2}{\partial\psi}=\frac{-ie^{-i\Phi(r,\psi)}}{2\gamma m\omega_y}F_\perp(\phi)L_0(r)\ , \tag{40}$$

with

$$L_0(r)=\frac{3}{2\pi\hat{\phi}^2}\left[1-\frac{r^2}{\hat{\phi}^2}\right]^{1/2} .$$

To solve Eq. (40), let

$$g_2(r,\psi)=\sum_m R_m(r)\exp(im\psi) . \quad (41)$$

Multiplying by $\exp(-i\mu\psi)$ and integrating over ψ yields

$$[Q(r)+\mu]R_\mu(r)=2\pi L_0(r)\int_0^{\hat{\phi}} x\,dx \sum_m R_m(x)\sum_n i\chi_n(\Omega)K_\mu(\tilde{n}r,\lambda r^2)K_m(\tilde{n}x,\lambda x^2) , \quad (42)$$

where

$$K_\mu(x,y)=e^{i\pi\mu/2}\sum_p J_{\mu+2p}(x)J_p(y) ,$$

$$\lambda=\frac{\Delta\omega_{SC}}{4\omega_s\hat{\phi}^2} ,$$

$$Q(r)=\frac{\omega_y-\Delta\omega_{SC}(1-r^2/2\hat{\phi}^2)-\Omega}{\omega_s} .$$

Next we expand the kernel in Eq. (42) as

$$\sum_n i\chi_n(\Omega)K_\mu(\tilde{n}r,\lambda r^2)K_m(\tilde{n}x,\lambda x^2) = \sum_{j,k} \alpha_{j,k}^{\mu,m} f_j^{|\mu|}(r)f_k^{|m|}(x) , \quad (43)$$

where $f_k^m(r)$ for $k=0,1,\ldots,$ form an orthornomal basis on $(0,\hat{\phi})$ with the weighting function $2\pi rL_0(r)$ for each superscript m. Polynomial expansion functions are given by

$$f_k^m(r)=\left[\frac{2^{m+2+1/2}}{3h_{k,m}}\right]^{1/2}\left[\frac{r}{\hat{\phi}}\right]^m P_k^{m,1/2}(1-2r^2/\hat{\phi}^2) , \quad (44)$$

where $P_k^{m,1/2}(x)$ is a Jacobi polynomial, and

$$h_{k,m}=\frac{2^{m+3/2}\Gamma(k+m+1)\Gamma(k+\frac{3}{2})}{(2k+m+\frac{3}{2})k!\Gamma(k+m+\frac{3}{2})} .$$

Defining

$$b_{k,m}=\int_0^{\hat{\phi}} x\,dx R_m(x)f_k^{|m|}(x)$$

yields a matrix equation given by

$$b_{p,\mu}=\sum_{m,k} b_{k,m}\sum_j \alpha_{j,k}^{\mu,m}\int_0^{\hat{\phi}} r\,dr\frac{2\pi L_0(r)f_p^{|\mu|}(r)f_j^{|\mu|}(r)}{Q(r)+\mu} . \quad (45)$$

Equation (45) simplifies in the low-intensity limit, where $K_m(\tilde{n}r,\lambda r^2)\approx e^{im\pi/2}J_m(\tilde{n}r)$. For this case, the coefficients in the expansion of the kernel are given by

$$\alpha_{j,k}^{\mu,m}=\sum_n i\chi_n i^{|\mu|-|m|}\tilde{f}_j^{|\mu|}(\tilde{n}\hat{\phi})\tilde{f}_k^{|m|}(\tilde{n}\hat{\phi}) , \quad (46)$$

where

$$\tilde{f}_k^m(x)=\frac{\Gamma(k+\frac{3}{2})}{k!}\left[\frac{2^{m+4+1/2}3}{\pi h_{k,m}}\right]^{1/2}\frac{j_{m+2k+1}(x)}{x} , \quad (47)$$

and $j_n(x)$ is the spherical Bessel function. In practice, the sum in Eq. (45) need to be truncated. Consider the simplest case where only the $p=\mu=0$ mode is included. Additionally, we will use Eq. (46). This leads to a dispersion relation of the form

$$1=\hat{\alpha}\int_0^1 \frac{3x\,dx\sqrt{1-x^2}}{-x^2+2[\Omega-\omega_\perp(0)]/\Delta\omega_{SC}} , \quad (48)$$

$$\hat{\alpha}=\frac{8}{5}\left|1-\frac{a^2}{b^2}\right|-\frac{2\omega_s}{\Delta\omega_{SC}}\sum_n i\hat{\chi}_n(\Omega)\left|\frac{3j_1(\tilde{n}\hat{\phi})}{\tilde{n}\hat{\phi}}\right|^2 , \quad (49)$$

where a is the beam radius, b is the pipe radius, and $\hat{\chi}_n(\Omega)$ is $\chi_n(\Omega)$ calculated without the space-charge contribution. A stability diagram for this dispersion relation is shown in Fig. 5. The system is stable if $\hat{\alpha}$ lies to the left of the curve.

The technique used above is also applicable to a line density that is constant within the bunch. This problem has been well studied [1,8] but the following equations are in a form that is suitable for solutions using numerical libraries.

The radial modes for $L_0(r)\propto(\hat{\phi}^2-r^2)^{-1/2}$ are $f_k^m(r)\propto r^m P_k^{m,-1/2}(1-2r^2/\hat{\phi}^2)$ [8]. The Vlasov equation is reduced to a matrix equation for the $b_{k,m}$s as in the previous case. The matrix is given by

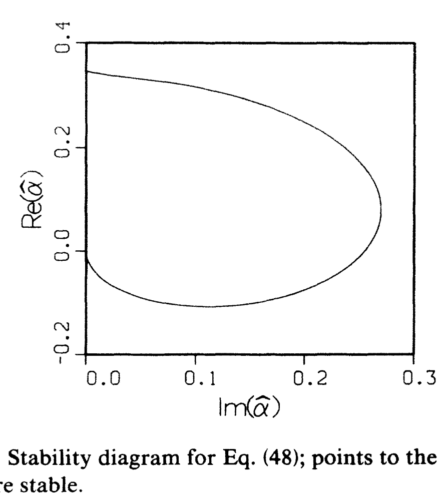

FIG. 5. Stability diagram for Eq. (48); points to the left of the contour are stable.

$$\left[\frac{\Omega-\omega_y+\Delta\omega_{\mathrm{SC}}}{\omega_s}\right]b_{p,q}=qb_{p,q}-i\sum_{k,m}b_{k,m}\sum_l \chi_l i^{|q|-|m|}g_p^{|q|}(\tilde{l}\hat{\phi})g_k^{|m|}(\tilde{l}\hat{\phi}) , \tag{50}$$

where

$$g_k^m(x)=\left[\frac{2(m+2k+\frac{1}{2})\Gamma(k+\frac{1}{2})\Gamma(m+k+\frac{1}{2})}{\pi\Gamma(k+1)\Gamma(k+m)}\right]^{1/2} j_{m+2k}(x) . \tag{51}$$

For χ_l=constant, the sum over l in Eq. (50) is zero unless $|m|+2k=|q|+2p$ [8]. For a given value of $|m|+2k$, there are a finite number of equations and the result is exact. For the general case, the matrix form of Eq. (50) is easily solved using numerical methods.

B. Gaussian line density

When the space-charge tune spread is larger than the synchrotron tune, several synchrotron modes can couple. Solving Eq. (45) is quite difficult, so we will present a different approach that is easier to program. Consider Eq. (37) with a Gaussian line density. We make the substitution $g_1=g_2\exp[iQ_y(\xi/\eta-1)\phi]$ and express the result in normalized, Cartesian variables $z=\phi/\sqrt{2}\sigma$, $v=\dot{\phi}/\sqrt{2}\sigma\omega_s$. This yields

$$-Qg_2(z,v)+i\left\{v\frac{\partial g_2}{\partial z}-z\frac{\partial g_2}{\partial v}\right\}+\frac{\Delta\omega_{\mathrm{SC}}}{\omega_s}\left\{g_2e^{-z^2}-(1-a^2/b^2)\frac{e^{-(z^2+v^2)}}{\sqrt{\pi}}\int_{-\infty}^{\infty}dv_1g_2(z,v_1)\right\}$$
$$=-i\frac{e^{-(z^2+v^2)}}{\pi}\sum_k\hat{\chi}_k e^{i\tilde{k}\sqrt{2}\sigma z}\int dz_1dv_1e^{-i\tilde{k}\sqrt{2}\sigma z_1}g_2(z_1,v_1) , \tag{52}$$

where $\tilde{k}=kM+s+Q_y(1-\xi/\eta)$, $\hat{\chi}_k$ is χ_k calculated without the space-charge contribution, a is the beam radius, b is the pipe radius, and $Q=(\omega_y-\Omega)/\omega_s$ as before. To solve Eq. (52), expand g_2 as

$$g_2(z,v)=\sum_{n,m}a_{n,m}H_n(z)H_m(v)e^{-(z^2+v^2)} , \tag{53}$$

where

$$H_m(x)=e^{x^2}\left[\frac{d}{dx}\right]^m e^{-x^2}$$

is the Hermite polynomial of order m, and the sum is over all pairs of non-negative integers. Substituting Eq. (53) in Eq. (52), multiplying by $H_p(z)H_q(v)dz\,dv$, and integrating yields a matrix for the expansion coefficients

$$Qa_{p,q}=\sum_{n,m}T_{p,q,n,m}a_{n,m} . \tag{54}$$

The matrix element is given by

$$T_{p,q,n,m}=i\{n\delta_{m+1}^q\delta_{n-1}^p-m\delta_{m-1}^q\delta_{n+1}^p\}+\frac{\Delta\omega_{\mathrm{SC}}}{\omega_s}\frac{\delta_m^q[1-\delta_m^0(1-a^2/b^2)]}{p!\sqrt{2\pi}}\left[\frac{i}{\sqrt{2}}\right]^{p-n}\left[\frac{n+p+1}{2}\right]\mathrm{even}(n+p)$$
$$+i\frac{\delta_m^0\delta_q^0}{p!}\left[\frac{i}{\sqrt{2}}\right]^{p-n}\sum_k\hat{\chi}_k(\sigma\tilde{k})^{n+p}e^{-\tilde{k}^2\sigma^2} , \tag{55}$$

where δ_n^m is the Kronecker δ, and even(k) is 1 when k is even and zero otherwise. In real machines, transverse instabilities are often expected and damping systems are installed. The effect of a linear feedback damping system can be included in the Vlasov equation and additional terms to the $T_{p,q,n,m}$ matrix are obtained. For a damper which integrates $D(t)$ over each bunch and produces a constant kick over the interval of each bunch, the addition to the matrix is

$$\delta T_{p,q,n,m}=\frac{i\alpha_{d,0}}{\omega_s}\delta_q^0\delta_m^0\frac{\bar{q}^{p+n}i^{p-n}}{p!2^p}e^{-\bar{q}^2/2} , \tag{56}$$

whether $\bar{q}=\sqrt{2}\sigma Q_y\xi/\eta$, and $\alpha_{d,0}$ is the damping rate for the rigid mode.

In practical applications, the sum in Eq. (54) needs to be truncated. We do this by taking terms with $0\le m+n\le l_{\max}$. When this is the case, and $\Delta\omega_{\mathrm{SC}}=0$, we find that Eq. (35) and (54) yield identical frequencies [11]. Conversely, any differences between the two must be due to the different ways of handling space charge.

As with Eq. (35), a computer code has been written to solve Eq. (54). For comparison purposes, consider the same broadband resonator as was used in Figs. 1–4. Plots of the real part of Q as a function of $\Delta\omega_{\mathrm{SC}}/\omega_s$ are

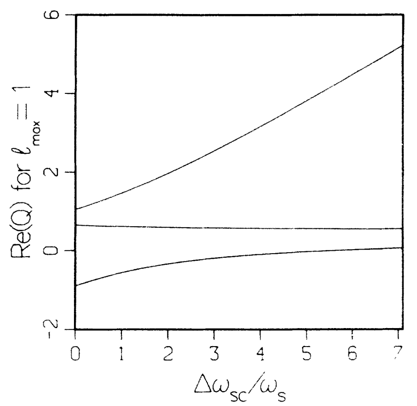

FIG. 6. Re(Q), for $l_{\max}=1$ as a function of $\Delta\omega_{SC}/\omega_s$.

shown in Figs. 6–9. Since space charge satisfies $\Delta\omega_{SC}/\omega_s=-\sqrt{2}f_0$, the limits on the horizontal axes are identical in Figs. 1–4 and Figs. 6–9. There are several differences between the two sets of figures.

One obvious difference is that Figs. 1–4 show a tendency for Re(Q) to decrease as the space-charge tune shift becomes larger, while the opposite is true for Figs. 6–9. This difference is largely due to the fact that the incoherent betatron tune was assumed constant for Figs. 1–4, while space-charge tune depression was included in the Hermite expansion. A closer agreement between the two methods could be found by taking the intensity dependence of the incoherent betatron tune into account for the moderate coupling case. However, this modification would have no effect on the growth rates predicted using Eq. (35). Another difference, which is not as apparent from the figures, is that the growth rates for the two cases were different. For the Hermite expansion, instability was not predicted until $l_{\max}=3$, and the peak growth rates for $l_{\max}=3$ and $l_{\max}=4$ were nearly identical. For $l_{\max}=5$, the peak growth rate was somewhat smaller.

As a second example, consider the resistive wall instability when the coherent space-charge tune shift is equal to the synchrotron tune. The growth rates of the most unstable mode as a function of $\Delta\omega_{SC}/\omega_s$ for various values of $l_{\max}$ are shown in Fig. 10. The three upper

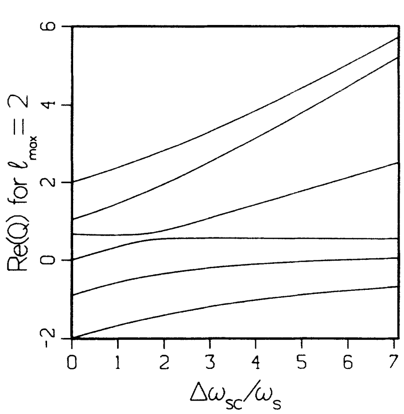

FIG. 7. Re(Q), for $l_{\max}=2$ as a function of $\Delta\omega_{SC}/\omega_s$.

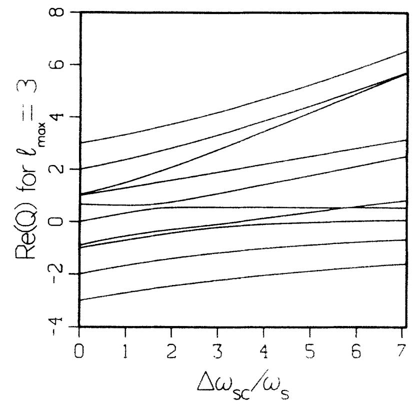

FIG. 8. Re(Q), for $l_{\max}=3$ as a function of $\Delta\omega_{SC}/\omega_s$.

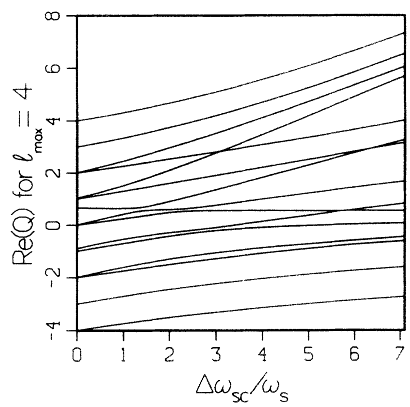

FIG. 9. Re(Q), for $l_{\max}=4$ as a function of $\Delta\omega_{SC}/\omega_s$.

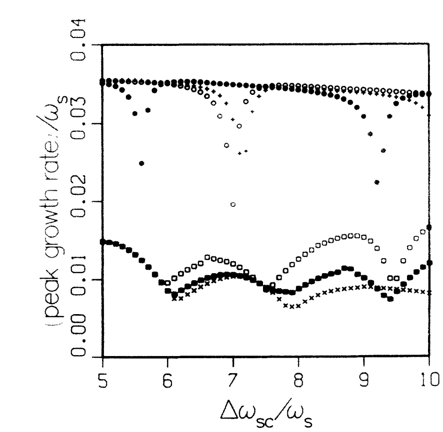

FIG. 10. Growth rate of the most unstable mode for a range of incoherent tune shifts. The calculations which include the incoherent space-charge tune spread for $l_{\max}=10$, 15, and 20 are denoted using ○, ●, and +, respectively. The calculations that ignore the incoherent space-charge tune spread for $l_{\max}=10$, 15, and 20 are denoted using □, ■, and ×, respectively.

curves were obtained using the Hermite expansion. The local minima are artifacts of the technique, but the least upper bound over the different $l_{\max}$ values is fairly smooth. They are also fairly close to the weak coupling limit value for the growth rate $=0.038\omega_s$. The three lower curves were obtained by ignoring the space-charge tune spread and suggest that the incoherent space-charge tune shift can lead to growth rate reduction. In Sec. III C, we consider an exactly soluble limit in which the incoherent forces can be handled exactly.

C. Strong coupling case

For the strong coupling case, $\Delta\omega_{\rm SC} \gg \omega_s$. Instead of trying to include the motion of a very large number of modes, we will neglect the synchrotron motion altogether, giving $\dot{\phi}=0$ and a beam that is frozen longitudinally. The vertical equation of motion is given by

$$\ddot{y}+\omega_y^2 y=\frac{F_\perp(x,y,\phi,t)}{\gamma m} , \qquad (57)$$

where both transverse coordinates appear in the collective force. The collective force contains contributions due to the currents flowing in the walls of the vacuum chamber and direct particle-particle forces that are responsible for the space-charge tuned spread. We average Eq. (57) over the particles at a fixed value of ϕ. The direct particle-particle forces cancel when averaged over the cross section, since they come from a Coulomb potential. We approximate the wall induced forces using the transverse impedance, as in the earlier sections. This results in an eigenvalue problem for $\langle y(\phi,t)\rangle =\langle y(\phi)\rangle \exp(-i\Omega t)$, which is given by

$$-\Omega^2\langle y(\phi)\rangle+\omega_y^2\langle y(\phi)\rangle=\frac{F_{\rm wall}(\phi)}{\gamma m} . \qquad (58)$$

The force due to the currents in the wall is given by Eq. (8) using space-charge impedance given by

$$Z_{\rm wall,SC}(\omega)=-i\frac{RZ_0}{b^2\beta^2\gamma^2} . \qquad (59)$$

To turn Eq. (58) into a proper eigenvalue problem, we approximate $\omega_y^2-\Omega^2\approx 2\omega_y(\omega_y-\Omega)$ and set $\Omega=\omega_y$ in the argument of the impedance. Multiplying the result by the line density gives

$$(\omega_y-\Omega)D(\phi)=\frac{\rho(\phi)}{2\omega_y\gamma m}F_{\rm wall}(\phi) , \qquad (60)$$

where $\rho(\phi)$ is the line density of the particles, and ω_y is the bare betatron frequency for an on-momentum particle. As a consistency check, note that setting $\omega_s=0$ in Eq. (37) leads to Eq. (60). We proceed to solve this equation for a model system.

Consider a system with space charge and a single narrow band resonator. It is assumed that the impedance of the resonator is negligible compared to space charge for all but a single value of $n=K$. Under these circumstances, Eq. (60) becomes

$$\delta\Omega D(\phi)=-\hat{\rho}(\phi)[\Delta D(\phi)+i\hat{R}D_K e^{i(KM+s)\phi}] , \qquad (61)$$

where $\delta\Omega=\Omega-\omega_y$, $-i\Delta/\hat{R}=Z_{\rm wall,SC}/R$, R is the resonator impedance when $n=K$, and $\hat{\rho}(\phi)$ is the normalized line density. Solving for $D(\phi)$ and extracting the Kth harmonic leads to a dispersion relation,

$$1=-\frac{iM\hat{R}}{2\pi}\int_{-\pi/M}^{\pi/M}\frac{\hat{\rho}(\phi)d\phi}{\delta\Omega+\Delta\hat{\rho}(\phi)} . \qquad (62)$$

For a constant line density with $\hat{\rho}(\phi)=1/2\hat{\phi}$ for $|\phi|\le\hat{\phi}$, the integral is trivial with the result $\delta\Omega=-iM\hat{R}/2\pi-\Delta/2\hat{\phi}$. The growth rate is independent of the reactance. On the other hand, if $\hat{\rho}(\phi)=\cos^2(\pi\phi/2\hat{\phi})/\hat{\phi}$, the frequency shift, calculated in Appendix B, is given by,

$$\delta\Omega=-\frac{\Delta}{\hat{\phi}}\left[1-\frac{\tilde{R}^2}{1+2i\tilde{R}}\right] , \qquad (63)$$

where $\tilde{R}=M\hat{R}\hat{\phi}/\pi\Delta$, and the solution is valid for $|1+2i\tilde{R}|>1$. No solution exits when $|1+2i\tilde{R}|<1$ and the system is Landau damped. Other smooth bunch shapes give similar results. Setting $\Delta=0$ gives $\text{Im}(\delta\Omega_0)=-M\hat{R}/2\pi$. The actual growth rate is always smaller than the $\Delta=0$ growth rate with the ratio of the actual growth rate to the $\Delta=0$ growth rate given by $\text{Im}(\delta\Omega)/\text{Im}(\delta\Omega_0)\lesssim 2|\tilde{R}|$. The growth rate reduction can be understood by considering the dipole eigenfunction

$$D(\phi)=\frac{\cos^2(\pi\phi/2\hat{\phi})e^{i(Km+s)\phi}}{\sin^2(\pi\phi/2\hat{\phi})-\tilde{R}^2/(1+2i\tilde{R})} .$$

For small $\tilde{R}$, the magnitude of D is relatively small unless $|\phi|\lesssim\hat{\phi}|\tilde{R}|$. In physical terms, the space-charge impedance causes the local coherent betatron frequency to vary along the bunch like a collection of oscillators with different natural frequencies. The resonator is only partially effective at maintaining the oscillators at a single coherent frequency, resulting in a reduced growth rate.

IV. CONCLUSIONS

The effect of space charge on transverse instabilities has been considered. In the low-intensity limit we found that incoherent space-charge tune spread can Landau damp transverse instabilities. However, this damping will occur only if the inductive part of the effective transverse impedance nearly cancels the space-charge impedance. In the moderate coupling regime, with a Gaussian line density, an expansion using Hermite polynomials has been used. This method was compared with earlier techniques that do not include the incoherent space-charge frequency spread, and it appears that including the frequency spread is needed for accurate results. In the high-intensity limit, where the synchrotron tune is small compared to the coherent space-charge frequency shift, we found that the incoherent space-charge forces are irrelevant. In this limit, the coherent space-charge forces, due to image currents on the vacuum chamber walls, can lead to growth rates which can be much smaller than those predicted using the weak coupling formulas.

APPENDIX A

In this appendix, it is shown that the growth rates obtained from Eq. (34) are bounded via $|\mathrm{Im}(Q)| \le |f_1|$, when f_0, f_2, and f_1^2 are all real. Rewrite Eq. (34) to yield

$$(Q-f_0)(Q-F_+)(Q-F_-)=Qf_1^2 , \tag{A1}$$

where $F_{\pm}=f_2/2\pm\sqrt{1+f_2^2/4}$. Notice that $F_+>0$ and $F_-<0$. Taking absolute values and using the triangle inequality yields

$$|\mathrm{Im}(Q)|^2 \le |Q-f_0||Q-F_+| = \frac{|Qf_1^2|}{|Q-F_-|} . \tag{A2}$$

Dividing by $Q-F_+$ instead of $Q-F_-$ yields

$$|\mathrm{Im}(Q)|^2 \le |Q-f_0||Q-F_-| = \frac{|Qf_1^2|}{|Q-F_+|} . \tag{A3}$$

Since $a \le b$ and $a \le c$ implies $a \le \min(b,c)$,

$$|\mathrm{Im}(Q)|^2 \le |f_1^2| \min\left[\frac{|Q|}{|Q-F_-|},\frac{|Q|}{|Q-F_+|}\right] . \tag{A4}$$

F_- and F_+ are real and of opposite sign. When they are viewed as vectors on the complex plane, one of them will add constructively to any Q one cares to choose. Hence, one of the denominators on the right-hand side will be larger than $|Q|$, giving $|\mathrm{Im}(Q)| \le |f_1|$.

APPENDIX B

The coherent frequency for a cosine squared line density is derived. Defining $\tilde{Q}=2\hat{\phi}\delta\Omega/\Delta$, Eq. (62) becomes

$$1=-\frac{i\tilde{R}}{2\pi}\int_{-\pi}^{\pi} d\theta \frac{1+\cos\theta}{\tilde{Q}+1+\cos\theta} . \tag{B1}$$

Making the substitution $z=\exp(i\theta)$, Eq. (B1) becomes

$$\frac{i}{\tilde{R}}=1+\frac{i\tilde{Q}}{\pi}\oint \frac{dz}{2z(1+\tilde{Q})+z^2+1} , \tag{B2}$$

where the integral is along the unit circle in the complex plane. The integrand of Eq. (B2) has simple poles at $z_{\pm}=-(1+\tilde{Q})\pm\sqrt{(1+\tilde{Q})^2-1}$. Since $z_+z_-=1$, either both poles lie on the unit circle or one lies inside and the other lies outside. Assume the second case, which is equivalent to assuming a nonsingular integrand in Eq. (B1). Define z_{in} to be the root inside the unit circle and z_{out} to be the root outside the unit circle. The integral is trivial with the result

$$1-\frac{i}{\tilde{R}}=\frac{2\tilde{Q}}{z_{\mathrm{in}}-z_{\mathrm{out}}} . \tag{B3}$$

Squaring both sides of Eq. (B3) gives

$$\left[1-\frac{i}{\tilde{R}}\right]^2=\frac{\tilde{Q}^2}{\tilde{Q}^2+2\tilde{Q}} . \tag{B4}$$

Assuming $\tilde{Q}\ne 0$ yields

$$\tilde{Q}=-2\frac{(1+i\tilde{R})^2}{1+2i\tilde{R}} , \tag{B5}$$

which is equivalent to Eq. (63). Equation (B5) suggests that very large frequency shifts are possible if $\tilde{R}\approx i/2$, but this does not occur. For $|1+2i\tilde{R}|<1$, Eq. (B4) is satisfied but Eq. (B3) is not, and the system is Landau damped.

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